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NOTES ON THE THEORIES OF JUPITER AND SATURN.

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ON account of their large masses and the near approach to commensurability of their mean motions, Jupiter and Saturn offer the most interesting, as well as the most difficult, field for research, in the planetary perturbations of the solar system. In the following remarks, without treating the subject in a complete manner, which would be impossible here, I intend only to point out a method of procedure and give a few illustrations of its use.

At present we shall notice only the introaction of the sun, Jupiter and Saturn. It will facilitate matters much if we employ differential equations in which the potential function is the same for both planets. This is accomplished by an orthogonal transformation of variables. Let us suppose that the coordinates of the sun in space are denoted by

$$X,\ Y \text{ and } Z,$$

those of Jupiter by

$$X + x + \varkappa x',\ Y + y + \varkappa y' \text{ and } Z + z + \varkappa z',$$

and those of Saturn by

$$X + x' + \varkappa x\ ,\ Y + y' + \varkappa y \text{ and } Z + z' + \varkappa z\ ,$$

where $\varkappa$ is a small constant to be so determined that the variables x, x', &c., may be orthogonal.

M, m and m' denoting severally the masses of the sun, Jupiter and Saturn, the *vis viva* T of the system is represented by the equation

$$2Tdt = MdX^2 + m(dX + dx + \varkappa dx')^2 + m'(dX + dx' + \varkappa dx)^2$$
$$+ \text{ similar terms in } Y, y, y' \text{ and } Z, z, z'$$

$$=(M+m+m')\left(dX+\frac{m+\varkappa m'}{M+m+m'}dx+\frac{m'+\varkappa m}{M+m+m'}dx'\right)^2$$

$$+\left(m+\varkappa^2 m'-\frac{(m+\varkappa m')^2}{M+m+m'}\right)dx^2$$

$$+\left(m'+\varkappa^2 m-\frac{(m'+\varkappa m)^2}{M+m+m'}\right)dx'^2$$

$$+2\left(\varkappa(m+m')-\frac{(m+\varkappa m')(m'+\varkappa m)}{M+m+m'}\right)dx\,dx'$$

$$+ \text{ similar terms in } Y, y, y', Z, z, z'.$$

In order that the system of variables may be orthogonal, the coefficient of $dxdx'$ in this expression must vanish, which gives us, for the determination of $\varkappa$, the quadratic equation

$$\varkappa^2-\left(\frac{M}{m}+\frac{M}{m'}+2\right)\varkappa+1=0.$$

Of this the smaller root must be taken. Employing Bessel's values of the masses of Jupiter and Saturn, $\frac{M}{m}=1047.879$, $\frac{M}{m'}=3501.6$. Hence

$$\varkappa^2-4551.479\varkappa+1=0,$$

whence we get

$$\varkappa=\frac{1}{4551.479}+\left(\frac{1}{4551.479}\right)^3+\ldots=0.0002197088.$$

For brevity we will put

$$\mu=m+\varkappa^2 m'-\frac{(m+\varkappa m')^2}{M+m+m'},$$

$$\mu'=m'+\varkappa^2 m-\frac{(m'+\varkappa m)^2}{M+m+m'}.$$

When the numerical values are substituted these equations give

$$\mu=0.9990467623\,m,\qquad \mu'=0.9997145123\,m'.$$

As we do not wish to know X, Y and Z, but only the six variables x, y, z, x', y' and z' which assign the position of Jupiter and Saturn relatively to the sun, we can altogether neglect the first term in the last expression for T, and write

$$T=\mu\frac{dx^2+dy^2+dz^2}{2dt^2}+\mu'\frac{dx'^2+dy'^2+dz'^2}{2dt^2}.$$

If we put

$$x^2+y^2+z^2=r^2,\quad x'^3+y'^2+z'^2=r'^2,\quad xx'+yy'+zz'=rr's,$$

the expression of the potential function is

$$\Omega = \frac{Mm}{[r^2+2\varkappa rr's+\varkappa^2r'^2]^{\frac{1}{2}}} + \frac{Mm'}{[r'^2+2\varkappa rr's+\varkappa^2r^2]^{\frac{1}{2}}} + \frac{mm'}{1-\varkappa}\frac{1}{[r'^2-2rr's+r^2]^{\frac{1}{2}}}.$$

From the last term it will be seen that the motion of two planets, whose coordinates are severally x, y, z and x', y', z', relatively to each other, is homothetic with the relative motion of Jupiter and Saturn. The differential equations of motion are

$$\frac{d^2x}{dt^2} = \frac{1}{\mu}\frac{d\Omega}{dx}, \qquad \frac{d^2y}{dt^2} = \frac{1}{\mu}\frac{d\Omega}{dy}, \qquad \frac{d^2z}{dt^2} = \frac{1}{\mu}\frac{d\Omega}{dz},$$

$$\frac{d^2x'}{dt^2} = \frac{1}{\mu'}\frac{d\Omega}{dx'}, \qquad \frac{d^2y'}{dt^2} = \frac{1}{\mu'}\frac{d\Omega}{dy'}, \qquad \frac{d^2z'}{dt^2} = \frac{1}{\mu'}\frac{d\Omega}{dz'}.$$

The first two radicals in Ω may be expanded in series proceeding according to ascending powers of $\varkappa$; and, since this constant is so small, the cube and higher powers of it may be neglected. Thus

$$\Omega = \frac{Mm}{r} + \frac{Mm'}{r'} - \varkappa Mm\frac{r'}{r^2}s - \varkappa Mm'\frac{r}{r'^2}s + \tfrac{1}{2}\varkappa^2 Mm\frac{r'^2}{r^3}(1+3s^2)$$

$$+ \tfrac{1}{2}\varkappa^2 Mm'\frac{r^2}{r'^3}(1+3s^2) + \frac{mm'}{1-\varkappa}\frac{1}{[r'^2-2rr's+r^2]^{\frac{1}{2}}}.$$

If for Ω are substituted only the first two terms of this expression, the differential equations are easily integrated, and the variables x, y, z and x', y', z' represent the motion of two planets moving according to the laws of elliptic motion, whose mean motions are

$$\sqrt{\frac{Mm}{\mu a^3}}, \qquad \sqrt{\frac{Mm'}{\mu' a'^3}}.$$

In terms of symbols whose meaning is well known, we will put

$$\begin{aligned} L &= \sqrt{[Mm\mu a]}, & L' &= \sqrt{[Mm'\mu' a']}, \\ G &= \sqrt{[Mm\mu a(1-e^2)]}, & G' &= \sqrt{[Mm'\mu' a'(1-e'^2)]}, \\ H &= \sqrt{[Mm\mu a(1-e^2)]}\cos i, & H' &= \sqrt{[Mm'\mu' a'(1-e'^2)]}\cos i, \end{aligned}$$

and denote the mean anomalies by l and l', the distances of the perihelia from the nodes by g and g' and the longitudes of the nodes by h and h', and, moreover, put

$$R = \frac{Mm}{2a} + \frac{Mm'}{2a'} - \varkappa M\left(m\frac{r'}{r^2} + m'\frac{r}{r'^2}\right)s + \tfrac{1}{2}\varkappa^2 M\left(m\frac{r'^2}{r^3} + m'\frac{r^2}{r'^3}\right)(1+3s^2)$$

$$+ \frac{mm'}{1-\varkappa}\frac{1}{[r'^2-2rr's+r^2]^{\frac{1}{2}}}.$$

We have then the following system of differential equations for determining the elements L, G, H, L', G', H', l, g, h, l', g', h':—

$$\frac{dL}{dt}=\frac{dR}{dl},\quad \frac{dl}{dt}=-\frac{dR}{dL},\quad \frac{dL'}{dt}=\frac{dR}{dl'},\quad \frac{dl'}{dt}=-\frac{dR}{dL'},$$

$$\frac{dG}{dt}=\frac{dR}{dg},\quad \frac{dg}{dt}=-\frac{dR}{dG},\quad \frac{dG'}{dt}=\frac{dR}{dg'},\quad \frac{dg'}{dt}=-\frac{dR}{dG'},$$

$$\frac{dH}{dt}=\frac{dR}{dh},\quad \frac{dh}{dt}=-\frac{dR}{dH},\quad \frac{dH'}{dt}=\frac{dR}{dh'},\quad \frac{dh'}{dt}=-\frac{dR}{dH'},$$

in which it is understood that R is expressed in terms of the elements L, G, H, &c.

As r is a function of the three elements L, G, l only, and r' of L', G', l' only, it follows that the six elements H, g, h, H', g' and h' enter in R only through s; hence we have the equations

$$\frac{dR}{dg}=\frac{dR}{ds}\frac{ds}{dg},\qquad \frac{dR}{dH}=\frac{dR}{ds}\frac{ds}{dH},\qquad \frac{dR}{dh}=\frac{dR}{ds}\frac{ds}{dh},$$

$$\frac{dR}{dg'}=\frac{dR}{ds}\frac{ds}{dg'},\qquad \frac{dR}{dH'}=\frac{dR}{ds}\frac{ds}{dH'},\qquad \frac{dR}{dh'}=\frac{dR}{ds}\frac{ds}{dh'}.$$

The expression for s being given by

$$rr's = xx' + yy' + zz',$$

and v and v' denoting the true anomalies, the rectangular coordinates have the equivalents

$$\begin{aligned}
x &= r[\cos h \cos(v+g) - \cos i \sin h \sin(v+g)],\\
y &= r[\sin h \cos(v+g) + \cos i \cos h \sin(v+g)],\\
z &= r \sin i \ \sin(v+g),\\
x' &= r'[\cos h' \cos(v'+g') - \cos i' \sin h' \sin(v'+g')],\\
y' &= r'[\sin h' \cos(v'+g') + \cos i' \cos h' \sin(v'+g')],\\
z' &= r' \sin i' \ \sin(v'+g').
\end{aligned}$$

Whence the following expression for s:

$$\begin{aligned}
s = {} & \cos(h-h')\cos(v+g)\cos(v'+g') + \cos i \cos i' \cos(h-h')\sin(v+g)\sin(v'+g')\\
& + \cos i \sin(h-h')\cos(v+g)\sin(v'+g')\\
& - \cos i \sin(h-h')\sin(v+g)\cos(v'+g')\\
& + \sin i \sin i' \sin(v+g)\sin(v'+g').
\end{aligned}$$

Remembering that v and v' contain only the same elements as r and r', and that

$$\cos i = \frac{H}{G},\quad \sin i = \frac{\sqrt{(G^2-H^2)}}{G},\quad \cos i' = \frac{H'}{G'},\quad \sin i' = \frac{\sqrt{(G'^2-H'^2)}}{G'},$$

it will be found that

$$\frac{d}{dt}\left[\sqrt{(G^2-H^2)}\cos h + \sqrt{(G'^2-H'^2)}\cos h'\right] = 0,$$

$$\frac{d}{dt}\Big[\sqrt{(G^2-H^2)}\sin h+\sqrt{(C'^2-H'^2)}\sin h'\Big]=0,$$

$$\frac{d}{dt}\Big[H+H'\Big]\qquad\qquad\qquad\qquad=0.$$

Hence we have the following integrals of the differential equations,

$$\sqrt{(G^2-H^2)}\cos h+\sqrt{(G'^2-H'^2)}\cos h=\text{a constant},$$

$$\sqrt{(G^2-H^2)}\sin h+\sqrt{(G'^2-H'^2)}\sin h=\text{a constant},$$

$$H+H'=\text{a constant}.$$

These integrals may be employed to diminish the number of differential equations. Thus far the system of planes, to which x, y, z, &c. are referred, has been left indefinite: let us now assume that the plane of maximum areas, called by Laplace the invariable plane, is chosen for the plane of xy. In this case it is well known that the constants of the first two of the integrals, given above, become zero. Then we shall have

$$\sqrt{(G^2-H^2)}\cos h\ +\ \sqrt{(G'^2-H'^2)}\cos h'=0,$$

$$\sqrt{(G^2-H^2)}\sin h\ +\ \sqrt{(G'^2-H'^2)}\sin h'=0,$$

$$H+H'=c,$$

c being an arbitrary constant. But since i and i' are supposed contained between 0° and 180°, the radicals in these expressions must be taken positively. Consequently the equations are equivalent to

$$h'=h+180^\circ,\qquad H+H'=c,\qquad H-H'=\frac{G^2-G'^2}{c}.$$

These equations determine the values of the elements H, H' and h' in terms of the rest, and they may be used to eliminate them from R. Then it is plain, from the expression of s, given above, that h will also disappear from R, and we shall have

$$R=\text{function}\ (L,\ G,\ L',\ G',\ l,\ g,\ l',\ g'),$$

and s takes the much simpler form

$$s=-\cos(v-v'+g-g')+\frac{(G+G')^2-c^2}{2GG'}\sin(v+g)\sin(v'+g').$$

As to the partial derivatives of R with respect to L, L', l, l', g, g', they are evidently unchanged by this elimination of the elements H, H', h, h'. But $\left(\frac{dR}{dG}\right)$ and $\left(\frac{dR}{dG'}\right)$ denoting the derivatives of R on the supposition of its containing the elements H, H', h, h', we have

$$\left(\frac{dR}{dG}\right)=\frac{dR}{dG}-\frac{dR}{dH}\frac{dH}{dG}-\frac{dR}{dH'}\frac{dH'}{dG},$$

$$\left(\frac{dR}{dG'}\right)=\frac{dR}{dG'}-\frac{dR}{dH}\frac{dH}{dG'}-\frac{dR}{dH'}\frac{dH'}{dG'}.$$

But we also have

$$\frac{dR}{dH} - \frac{dR}{dH'} = \frac{d(h'-h)}{dt} = 0,$$

hence

$$\left(\frac{dR}{dG}\right) = \frac{dR}{dG} - \frac{dR}{dH}\frac{d(H+H')}{dG} = \frac{dR}{dG},$$

$$\left(\frac{dR}{dG'}\right) = \frac{dR}{dG'} - \frac{dR}{dH}\frac{d(H+H')}{dG'} = \frac{dR}{dG'}.$$

Moreover

$$\frac{dR}{dc} = \frac{dR}{dH}\frac{dH}{dc} + \frac{dR}{dH'}\frac{dH'}{dc} = \frac{dR}{dH}\frac{d(H+H')}{dc} = \frac{dR}{dH}.$$

Thus the system of differential equations still retains its canonical form, and is

$$\frac{dL}{dt} = \frac{dR}{dl}, \quad \frac{dL'}{dt} = \frac{dR}{dl'}, \quad \frac{dG}{dt} = \frac{dR}{dg}, \quad \frac{dG'}{dt} = \frac{dR}{dg'}.$$

$$\frac{dl}{dt} = -\frac{dR}{dL}, \quad \frac{dl'}{dt} = -\frac{dR}{dL'}, \quad \frac{dg}{dt} = -\frac{dR}{dG}, \quad \frac{dg'}{dt} = -\frac{dR}{dG'}.$$

After this system of eight differential equations is integrated, the value of h is found by a quadrature from the equation

$$\frac{dh}{dt} = -\frac{dR}{dc}.$$

These integrations introduce nine arbitrary constants which, together with c, make ten. The reference of the coordinates to any arbitrary planes introduces three more, but one of these coalesces with the constant which completes the value of h.

The time t does not explicitly enter R, hence the complete derivative of it with respect to t is

$$\frac{dR}{dt} = \frac{dR}{dL}\frac{dL}{dt} + \frac{dR}{dl}\frac{dl}{dt} + \text{\&c.}.$$

If, in this, are substituted the values of $\frac{dL}{dt}, \frac{dl}{dt}$, &c.., from the equations just given, we shall find that it vanishes; hence

$$R = \text{a constant}$$

is an integral of the system of differential equations. This integral may be employed to eliminate one of the elements, as L, from the equations. We can also take one of the elements, as l, for the independent variable in place of t. The system of equations, to be integrated, is then reduced to the six

$$\frac{dL'}{dl}=-\frac{\frac{dR}{dl'}}{\frac{dR}{dL}},\qquad \frac{dG}{dl}=-\frac{\frac{dR}{dg}}{\frac{dR}{dL}},\qquad \frac{dG'}{dl}=-\frac{\frac{dR}{dg'}}{\frac{dR}{dL}},$$

$$\frac{dl'}{dl}=\frac{\frac{dR}{dL'}}{\frac{dR}{dL}},\qquad \frac{dg}{dl}=\frac{\frac{dR}{dG}}{\frac{dR}{dL}},\qquad \frac{dg'}{dl}=\frac{\frac{dR}{dG'}}{\frac{dR}{dL}}.$$

A simpler form can be given to them. If the solution of $R=$ a constant gives

$$L = \text{function } (L',\ G,\ G',\ l',\ g,\ g',\ l),$$

and L is supposed to stand for the right member of this, the foregoing equations can be written

$$\frac{dL'}{dl}=\frac{dL}{dl'},\qquad \frac{dG}{dl}=\frac{dL}{dg},\qquad \frac{dG'}{dl}=\frac{dL}{dg'},$$

$$\frac{dl'}{dl}=-\frac{dL}{dL'},\qquad \frac{dg}{dl}=-\frac{dL}{dG},\qquad \frac{dg'}{dl}=-\frac{dL}{dG'}.$$

When the values of L', G, G', l', g and g' in terms of l have been derived from the integrals of these, they can be substituted in the equation $\frac{dl}{dt}=-\frac{dR}{dt}$, which will then give t in terms of l, by a quadrature. By inverting this we shall have l in terms of t; and by substituting this in equations previously obtained we shall have the values of all the other elements in terms of t.

It will be noticed that R is a homogeneous function of L, L', G, G' and c of the degree —2; hence we shall have

$$L\frac{dR}{dL}+L'\frac{dR}{dL'}+G\frac{dR}{dG}+G'\frac{dR}{dG'}+c\frac{dR}{dc}=-2R=\text{a const.}$$

and, as a consequence of this,

$$L\frac{dl}{dt}+L'\frac{dl'}{dt}+G\frac{dg}{dt}+G'\frac{dg'}{dt}+c\frac{dh}{dt}=\quad 2R=\text{a const.}$$

Thus, if the rate of motion of each elementary argument l, l' &c., be multiplied by the element which is conjugate to it, the sum of the products is invariable.

The sines of half the inclinations of the orbits on the plane of maximum areas are

$$\sin\frac{i}{2} = \sqrt{\left[\frac{(G+G'-c)(G'-G+c)}{4cG}\right]},$$

$$\sin\frac{i'}{2} = \sqrt{\left[\frac{(G'+G-c)(G-G'+c)}{4cG'}\right]}.$$

Thus, in the special case where the two planets move in the same plane, we have

$$G + G' = c.$$

This equation may be employed to eliminate one of the elements G or G' from R. In the same case, the expression for s is reduced to

$$s = -\cos(v - v' + g - g').$$

Then, if we put

$$G - G' = \Gamma, \qquad g - g' = \gamma,$$

R will be a function of L, L', Γ, l, l', γ, and we shall have, for determining these variables, the system of differential equations

$$\frac{dL}{dt} = \frac{dR}{dl}, \qquad \frac{dL'}{dt} = \frac{dR}{dl'}, \qquad \frac{d\Gamma}{dt} = \frac{dR}{d\gamma},$$

$$\frac{dl}{dt} = -\frac{dR}{dL}, \qquad \frac{dl'}{dt} = -\frac{dR}{dL'}, \qquad \frac{d\gamma}{dt} = -\frac{dR}{d\Gamma}.$$

After these are integrated, the value of $g+g'$ will be got by a quadrature from the equation

$$\frac{d(g+g')}{dt} = -\frac{dR}{dc}.$$

If the value of L is obtained from the solution of R = a constant, and we have

$$L = \text{function}\ (L', \Gamma, l', \gamma, l),$$

and l is adopted as the independent variable in place of t, the solution of this special case is reduced to the integration of the four equations

$$\frac{dL'}{dl} = \frac{dL}{dl'}, \quad \frac{dl'}{dl} = -\frac{dL}{dL'}, \quad \frac{d\Gamma}{dl} = \frac{dL}{d\gamma}, \quad \frac{d\gamma}{dl} = -\frac{dL}{d\Gamma}.$$

The angle between the planes of the orbits of Jupiter and Saturn is about $1\frac{1}{4}°$. This is small enough to make the terms, which are multiplied by the square of the sine of half it, and which are besides of two or more dimensions with respect to disturbing forces, practically insignificant. Thus, while we are engaged in developing those terms of the coordinates which demand the highest degree of approximation relatively to disturbing forces, we shall assume that the planes coincide; the determination of the effect of non-coincidence of these planes being reserved to the end, when it will be always sufficient to limit ourselves to the first power of the disturbing force.

[To be continued.]